J. Fluid Mech. (1995), *vol.* 294, *pp.* 155–180

Complementary approximations to wave scattering by vertical barriers

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(Received 20 October and in revised form 28 February 1995)

Scattering of waves by vertical barriers in infinite-depth water has received much attention due to the ability to solve many of these problems exactly. However, the same problems in finite depth require the use of approximation methods. In this paper we present an accurate method of solving these problems based on a Galerkin approximation. We will show how highly accurate complementary bounds can be computed with relative ease for many scattering problems involving vertical barriers in finite depth and also for a sloshing problem involving a vertical barrier in a rectangular tank.

1. Introduction

Two-dimensional problems involving the scattering of water waves by thin vertical barriers in deep water, when described by classical linear water-wave theory, have the property that they can be solved analytically in closed form with the important features such as the reflection and transmission coefficients expressible in terms of known functions or definite integrals. Thus Ursell (1947) solved for the velocity field everywhere in the case of a thin vertical barrier immersed to a depth a beneath the surface of deep water, in the presence of incident waves of wavelength $\lambda(= 2\pi/k)$. In particular he showed that the modulus of the ratio of reflected to incident wave elevation, $|R|$, could be expressed in terms of modified Bessel functions of argument ka. Earlier, Dean (1945) had obtained similar results for the scattering of an incident wave by a barrier extending downwards from a point a distance a below the free surface.

Subsequently other authors, notably Porter (1974), Mei (1966) and Lewin (1963) showed that the scattering of an incident wave by any number of in-line vertical barriers with arbitrary gaps between them, in deep water, could, in principle, be solved in closed form. Evans (1970) obtained such a solution for the scattering of an incident wave by a totally submerged vertical barrier. For two identical parallel vertical surface-piercing barriers, Levine & Rodemich (1958) showed that in this case also an explicit expression was possible although they did not obtain the detailed solution. Jarvis (1971) solved the complementary problem of two identical semi-infinite barriers extending indefinitely downwards from a point a beneath the surface and he obtained curves of reflection and transmission coefficients as a function of frequency, barrier spacing, and depth of submergence of the barriers, showing how for particular configurations, $|R| = 0$. In contrast to these exact results recently Parsons & Martin (1994) have shown how numerical results for thin arbitrarily oriented straight barriers or curved barriers can be obtained by consideration of the solutions of certain hypersingular integral equations.

The reason for the explicit nature of solutions for two-dimensional vertical-barrier problems in infinitely deep water is as follows. Solutions of the two-dimensional Laplace equation can be found amongst complex analytic functions, and for vertical thin barriers the condition of no flow across the barriers and the mixed free-surface condition can together be expressed as conditions on the real and imaginary parts of a single new so-called reduced complex potential being a simple combination of the original complex potential and its derivative. The problem is thereby reduced to finding a single simpler complex function satisfying certain conditions and having certain singularities followed by solving a first-order ordinary differential equation for the original potential. Although straightforward in principle, the details can be complicated, no more so than in a generalization of the technique provided by John (1948) in considering the scattering by a surface-piercing barrier making an angle $\pi/2n$ (n an integer) to the horizontal.

Because of the difficulties in the technique and also because the method fails for obliquely incident waves where complex function theory is no longer applicable, many authors have sought good approximation techniques. Thus Evans & Morris (1972) obtained good complementary bounds for the reflection coefficient for obliquely incident waves on a surface-piercing barrier in infinitely deep water. They set up the problem in two distinct ways resulting in singular integral equations for both the horizontal velocity across the gap below the barrier and also the potential difference across the barrier. By using the exact result of Ursell (1947) for normally incident waves as a one-term variational approximation, they obtained accurate upper and lower bounds in the form of a definite integral for the reflection coefficient. By repeating the technique for the two surface-piercing barriers considered by Levine & Rodemich (1958) they were able to show that in normally incident waves, for certain values of the spacing and length of the barriers, and incident wave frequency, the transmission coefficient vanished, whilst for other values the reflection coefficient vanished. The possibility of wave-interference effects between two or more bodies producing a zero reflection so that the bodies appeared transparent to the waves – apart from a phase change – was well-known but this seemed to be the first indication that combinations of partially immersed bodies could block the waves completely. Confirmation of this result was provided by Newman (1974) who used a matched asymptotic expansion approach to consider the scattering by two closely spaced barriers in infinitely deep water and confirmed the predictions of Evans & Morris (1972), and McIver (1985) who considered the general problem of two arbitrarily spaced surface-piercing barriers in finite depth.

With the exception of the work of McIver (1985) all the above-mentioned problems were in infinitely deep water and involved either explicit solutions or approximate solutions. McIver (1985) used matched eigenfunction expansions and orthogonal expansions involving trigonometric functions to obtain an infinite system of equations with an infinite number of unknowns. A similar approach for finite water depth was used by Smith (1983) for the scattering of an incident wave by a single surface-piercing barrier, and repeated by C.M. Linton (personal communication). The convergence of these solutions is slow and the accuracy of the results open to question. Thus even for the simplest problem of the single barrier in finite depth, Linton needed to solve a 400×400 system to obtain results which now turn out to have only two-figure accuracy.

The aim of the present work is to consider a number of vertical-barrier problems with gaps, using classical linear water-wave theory, and to provide a method which gives extremely accurate results for the reflection and transmission coefficients and other properties with minimum effort, based on deriving complementary bounds on quantities of interest. The problems to be considered include the scattering of an obliquely incident plane wave by either a single vertical barrier with an arbitrary number of gaps in finite-depth water, or a pair of identical vertical barriers where symmetry considerations enable us to reduce the problem to two simpler problems each involving a single barrier with gaps. We also consider the sloshing frequencies in a rectangular tank containing a plane vertical baffle with gaps parallel to one pair of sides. The two-dimensional version of this problem was considered by Evans & McIver (1987) using matched eigenfunction expansions.

In all of these problems the approach is the same, namely to derive an integral equation for a function $u(y)$ proportional to the horizontal fluid velocity in the gap interval L_g and to express an integral property of the solution over the gap in terms of a real quantity A having physical significance. Thus in operator notation, we shall obtain for each problem

$$Ku = \psi_0, \quad u \in L_g, \tag{1.1}$$

$$(u, \psi_0) = A, \tag{1.2}$$

where K is a real, linear, positive-definite, symmetric integral operator and (1.2) denotes the real inner product over L_g. A second integral equation will be derived for a function $p(y)$ proportional to the pressure jump across the interval occupied by the barrier, L_b. It will be found that this may be expressed as

$$Mp = \psi_0, \quad p \in L_b, \tag{1.3}$$

where, significantly

$$(p, \psi_0) = A^{-1}, \tag{1.4}$$

and again M is a real, linear, positive-definite, symmetric integral operator.

In each of these problems the main quantity of interest is A itself rather than $u(y)$ or $p(y)$.

The system (1.1) and (1.2) can be expressed in variational form by writing

$$A = (u, \psi_0)^2/(Ku, u), \tag{1.5}$$

when it is easily shown that $Ku = \psi_0$ is satisfied if and only if the expression (1.5) is stationary for small independent variations of u about its correct value. This forms the basis for the Schwinger variational method in which we write

$$u(y) = \sum_{n=0}^{N} a_n u_n(y), \tag{1.6}$$

and choose the a_n to make (1.5) stationary, the resulting value of A being a good approximation to the exact result. A similar approximation, namely

$$p(y) = \sum_{n=0}^{N} b_n p_n(y), \tag{1.7}$$

can be used to obtain an approximation to A from the variational form of (1.3), (1.4) which is

$$A = (Mp, p)/(p, \psi_0)^2. \tag{1.8}$$

Here we shall choose the alternative more direct Galerkin method which has been shown by Jones (1964, p.259) to be equivalent to the variational approach.

We substitute (1.6) into (1.1) multiply by $u_m(y)$ and integrate over L_g to obtain

$$\sum_{n=0}^{N} a_n K_{mn} = F_{m0}, \tag{1.9}$$

$$\sum_{n=0}^{N} a_n F_{n0} = A_l \simeq A, \tag{1.10}$$

where

$$K_{mn} = (Ku_n, u_m), \tag{1.11}$$

$$F_{mn} = (\psi_n, u_m). \tag{1.12}$$

Notice that by solving (1.9) for the a_n we have established an approximation U to u, given by (1.6), which satisfies

$$(U, KU) = (U, \psi_0) \tag{1.13}$$

rather than (1.1) and approximates A by (U, ψ_0). Now

$$\begin{aligned}(u, \psi_0) - (U, \psi_0) &= (u, \psi_0) - 2(U, \psi_0) + (U, \psi_0) \\ &= (u, Ku) - 2(U, Ku) + (U, KU) \\ &= (u - U, K(u - U)),\end{aligned} \tag{1.14}$$

since K is symmetric, whence

$$(U, \psi_0) \leqslant A, \tag{1.15}$$

since $(f, Kf) \geqslant 0$ for all f.

Similarly, we substitute (1.7) into (1.3), multiply by $p_m(y)$ and integrate over L_b to obtain

$$\sum_{n=0}^{N} b_n M_{mn} = G_{m0}, \tag{1.16}$$

$$\sum_{n=0}^{N} b_n G_{n0} = A_u^{-1} \simeq A^{-1}, \tag{1.17}$$

where

$$M_{mn} = (Mp_n, p_m), \quad G_{mn} = (\psi_n, p_m). \tag{1.18}$$

It follows immediately that the approximations A_l and A_u to A satisfy

$$A_l \leqslant A \leqslant A_u. \tag{1.19}$$

The success of the method depends upon a judicious choice of the functions $u_n(y)$ and $p_n(y)$ both to reflect the physical characteristics of the unknowns $u(y)$, $p(y)$ in the particular problem under consideration, and to provide a form for the K_{mn}, M_{mn} which can easily be computed. For the cases of just one gap and barrier, a single set of functions is sufficient; for two gaps or two barriers, two separate expansions are needed and the theory described requires slight modification.

In the next section we formulate the problem of the scattering of an obliquely incident plane wave in finite water depth h by a vertical barrier with gaps (see figure 1), and derive in detail the infinite system of equations for the a_n and b_n in (1.9), (1.16) for the separate cases of a single surface-piercing barrier and a submerged bottom-standing barrier, and provide reasons for the choice of $u_n(y)$

and $p_n(y)$. The case of a single gap with a barrier above and below it and the finite totally submerged vertical barrier are also considered. Here expansions in terms of two sets of functions are necessary for the pressure difference and velocity formulations respectively since the domains of the integral equations are disjoint in these cases.

In §3 the problem of scattering of an obliquely incident plane wave in water of depth h by two identical barriers a distance $2b$ apart with gaps is considered. By making use of symmetry arguments this problem can be replaced by two separate problems each involving a single vertical barrier with gaps, a distance b from a vertical wall on which either the potential or its normal derivative vanishes. The only modification is in the form of K_{mn}, M_{mn} in the infinite system and in the connection between A in each case and the overall reflection coefficient. The same sets of functions used in §2 are chosen to provide complementary bounds to A and hence estimates of $|R|$, $|T|$. Of particular interest in this problem is the existence of parameter values at which $|R|$ and $|T|$ vanish. McIver (1985) describes in detail the care required to determine these resonant frequencies accurately.

In the short §4 the problem of the sloshing frequencies of water in a rectangular tank containing a thin vertical barrier with gaps, parallel to a pair of sides, is considered. The modifications to the previous cases are slight and an expression is obtained from which the resonant frequencies can be determined.

Results for all these problems are presented in §5 together with a description of the computations involved. Thus in the case of the surface-piercing barrier and the submerged bottom-standing barrier, tables of A_l and A_u are presented for typical values of geometrical and wave parameters illustrating how accurate the method is in determining A for moderate values of N, the truncation parameter. Curves of $|R|$ and $|T|$ derived from A_l and A_u are also shown, as a function of non-dimensional wavenumber and a comparison made with the exact infinite-depth results of Ursell (1947) and Dean (1945). Also shown is the variation of $|R|$ and $|T|$ with the angle of incidence of the waves, for typical wavenumber and geometry. Similar accurate results for A_l, A_u are obtained in the cases of a gap in a barrier extending throughout the depth and a totally submerged barrier and typical curves of $|R|$, $|T|$ are presented.

The interesting phenomena of zero reflection and zero transmission occur in the problem of two surface-piercing barriers and a table is presented which illustrates the accuracy of the method in determining quantities corresponding to A for this problem from which zeros can easily be computed. Also shown is a curve, comparable to that given by McIver (1985), of the first few values of the zeros as a function of barrier spacing. Finally for the sloshing problem a table is presented showing again how accurately the value corresponding to A can be determined for a range of geometric parameters, from which the sloshing frequencies can be determined easily and accurately.

2. Scattering by a single barrier with gaps

Cartesian coordinates are chosen with the mean-free surface $y = 0$. The fluid, of depth h, occupies $0 < y < h$, $-\infty < x, z < \infty$ and the barrier with gaps occupies $x = 0$, $0 \leqslant y \leqslant h$, $-\infty < z < \infty$. We denote the barrier itself by L_b and the gaps by L_g so that $L_b \cup L_g$ is $[0, h]$.

It is assumed that a wave of frequency $\omega/2\pi$ is incident from $x = +\infty$ and is partially reflected at $x = 0$ and partially transmitted. The wave makes an angle θ

with the plane $z = 0$. Then it follows from linear water-wave theory that there exists a velocity potential $\Phi(x, y, z, t)$ and that we may write

$$\Phi(x, y, z, t) = \mathrm{Re}\,\phi(x, y)\mathrm{e}^{\mathrm{i}lz}\mathrm{e}^{-\mathrm{i}\omega t} \tag{2.1}$$

where l is the wavenumber in the z-direction. Then $\phi(x, y)$ satisfies

$$\phi_{xx} + \phi_{yy} - l^2\phi = 0, \quad 0 < y < h, \quad (x, y) \notin L_b, \tag{2.2}$$

$$\phi_y = 0, \quad y = h, \quad -\infty < x < \infty, \tag{2.3}$$

$$\phi_x = 0, \quad (x, y) \in L_b, \tag{2.4}$$

$$K\phi + \phi_y = 0, \quad y = 0, \quad (x, y) \notin L_b, \tag{2.5}$$

where $K = \omega^2/g$,

$$\phi,\ \phi_x \text{ are continuous for } (x, y) \in L_g. \tag{2.6}$$

Vertical eigenfunctions can be constructed which are orthonormal over $[0, h]$. Thus

$$\psi_n(y) = N_n^{-1/2} \cos k_n(h - y), \tag{2.7}$$

where

$$N_n = \frac{1}{2}\left(1 + \frac{\sin 2k_n h}{2k_n h}\right), \quad n \geqslant 0, \tag{2.8}$$

and where the k_n are the positive roots of

$$K + k_n \tan k_n h = 0, \quad n = 1, 2, \ldots \tag{2.9}$$

whilst $k_0 = -\mathrm{i}k$ and k is the positive root of

$$K = k \tanh kh. \tag{2.10}$$

Then the ψ_n satisfy

$$\frac{1}{h}\int_0^h \psi_n(y)\psi_m(y)\,\mathrm{d}y = \delta_{mn}, \tag{2.11}$$

where δ_{mn} is the Kronecker delta. The most general solutions of (2.2), (2.3), (2.5) which have the correct behaviour as $x \to \pm\infty$ are

$$\phi(x, y) = (\mathrm{e}^{-\mathrm{i}\alpha x} + R\mathrm{e}^{\mathrm{i}\alpha x})\psi_0(y) + \sum_{n=1}^{\infty} A_n \mathrm{e}^{-\alpha_n x}\psi_n(y), \quad \text{for } x > 0, \tag{2.12}$$

and

$$\phi(x, y) = T\mathrm{e}^{-\mathrm{i}\alpha x}\psi_0(y) + \sum_{n=1}^{\infty} B_n \mathrm{e}^{\alpha_n x}\psi_n(y), \quad \text{for } x < 0. \tag{2.13}$$

Here $\alpha = (k^2 - l^2)^{1/2} = k\cos\theta$, $l = k\sin\theta$, and $\alpha_n = (k_n^2 + l^2)^{1/2}$. It remains to determine R, T, A_n, B_n so that conditions (2.4) and (2.6) are satisfied. We define $\phi_x|_{x=0} \equiv U(y)$ and the jump in ϕ, $[\phi]_{x=0} \equiv P(y)$.

Then from (2.4) and (2.6)

$$U(y) = 0, \quad y \in L_b, \tag{2.14}$$

$$P(y) = 0, \quad y \in L_g, \tag{2.15}$$

Now from (2.6), (2.14) $U(y)$ is continuous for $y \in [0, h]$ so from (2.12), (2.13)

$$U(y) = -\mathrm{i}\alpha(1 - R)\psi_0(y) - \sum_{n=1}^{\infty} \alpha_n A_n \psi_n(y), \tag{2.16}$$

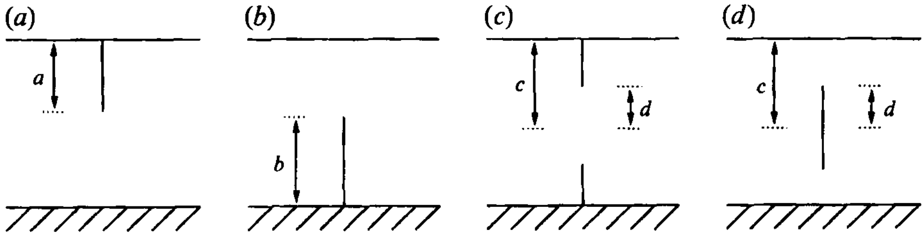

FIGURE 1. Definition sketches: (a) a surface-piercing barrier; (b) a bottom-standing barrier; (c) a barrier with a gap; (d) a totally submerged barrier.

$$= -\mathrm{i}\alpha T\psi_0(y) + \sum_{n=1}^{\infty} \alpha_n B_n \psi_n(y), \tag{2.17}$$

from which, after multiplying by $\psi_m(y)$ and integrating over $[0, h]$, we obtain

$$-\mathrm{i}\alpha(1-R) = -\mathrm{i}\alpha T = \frac{1}{h}\int_{L_g} U(y)\psi_0(y)\,\mathrm{d}y = U_0, \tag{2.18}$$

$$-\alpha_n A_n = \alpha_n B_n = \frac{1}{h}\int_{L_g} U(y)\psi_n(y)\,\mathrm{d}y = U_n, \tag{2.19}$$

where (2.14) has been used and

$$U(y) = \sum_{n=0}^{N} U_n\psi_n(y), \quad y \in [0, h]. \tag{2.20}$$

Application of (2.15) to (2.12), (2.13) now yields, after using (2.18), (2.19),

$$R\psi_0(y) = \int_{L_g} U(t)K(y,t)\,\mathrm{d}t, \quad y \in L_g, \tag{2.21}$$

where

$$K(y,t) = \sum_{n=1}^{\infty} (\alpha_n h)^{-1}\psi_n(y)\psi_n(t). \tag{2.22}$$

Defining $u(y)$ by $U(y) = Ru(y)$ and using (2.18) now gives

$$Ku \equiv \int_{L_g} u(t)K(y,t)\,\mathrm{d}t = \psi_0(y), \quad y \in L_g, \tag{2.23}$$

$$(u, \psi_0) \equiv \int_{L_g} u(y)\psi_0(y)\,\mathrm{d}y = A = -\mathrm{i}\alpha h(1-R)/R. \tag{2.24}$$

Notice that $(u, Kv) = (v, Ku)$ and $(u, Ku) \geqslant 0$, for all u, v, using (2.22) to (2.24). Notice also that it follows immediately from (2.24) that

$$R = \alpha h/(\alpha h + \mathrm{i}A), \tag{2.25}$$

whilst from (2.18)

$$T = \mathrm{i}A/(\alpha h + \mathrm{i}A), \tag{2.26}$$

and, since from (2.23), (2.24), A is real, $|R|^2 + |T|^2 = 1$ as expected from energy considerations.

We now assume we have just a single gap and, as described in the Introduction, we make the approximation (1.6) leading to the system (1.9), (1.10) where here

$$K_{mn} = (Ku_n, u_m) = \sum_{r=1}^{\infty}(\alpha_r h)^{-1} F_{mr} F_{nr}, \tag{2.27}$$

with

$$F_{mn} = (\psi_n, u_m) = \int_{L_g} u_m(y)\psi_n(y)\,\mathrm{d}y, \tag{2.28}$$

and (2.22) has been used. The resulting approximation $A_l \leqslant A$ is then given by (1.10). A similar integral equation for $P(y)$ can be obtained as follows. From (2.12), (2.13)

$$P(y) = 2R\psi_0(y) - 2\sum_{n=1}^{\infty}\alpha_n^{-1} U_n \psi_n(y), \quad y \in [0, h], \tag{2.29}$$

where (2.18), (2.19) have been used. Thus, multiplying by $\psi_m(y)$ and integrating over $[0, h]$

$$2R = \frac{1}{h}\int_{L_b} P(y)\psi_0(y)\,\mathrm{d}y = P_0, \tag{2.30}$$

$$-2\alpha_n^{-1} U_n = \frac{1}{h}\int_{L_b} P(y)\psi_n(y)\,\mathrm{d}y = P_n, \tag{2.31}$$

where (2.15) has been used, and where

$$P(y) = \sum_{n=1}^{\infty} P_n \psi_n(y), \quad y \in [0, h]. \tag{2.32}$$

Application of (2.14) to (2.16) now yields, for $y \in L_b$,

$$-\mathrm{i}\alpha(1 - R)\psi_0(y) = \sum_{n=1}^{\infty}\alpha_n A_n \psi_n(y)$$

$$= -\sum_{n=1}^{\infty} U_n \psi_n(y),$$

from (2.19),

$$= \sum_{n=1}^{\infty}\psi_n(y)\frac{\alpha_n}{2h}\int_{L_b} P(t)\psi_n(t)\,\mathrm{d}t, \tag{2.33}$$

from (2.31) or, writing $P(y) = -2\mathrm{i}h^2\alpha(1 - R)p(y)$,

$$\sum_{n=1}^{\infty}\alpha_n h\int_{L_b} p(t)\psi_n(t)\psi_n(y)\,\mathrm{d}t = \psi_0(y), \quad y \in L_b, \tag{2.34}$$

and from (2.30),

$$\int_{L_b} p(y)\psi_0(y)\,\mathrm{d}y = A^{-1}. \tag{2.35}$$

Notice that, unlike in the case of the velocity $u(y)$, we cannot immediately interchange the summation and integration in (2.34) to obtain (1.3), (1.4) since the resulting infinite sum is divergent. A recent study of the numerical solution of the hypersingular integral equation obtained if the limits are interchanged has been made by Parsons & Martin (1994). We can however continue to apply the Galerkin method as before, and, assuming initially that there is just a single barrier, we can choose an appropriate set of functions and expand $p(y)$ in the form

$$p(y) = \sum_{n=0}^{N} b_n p_n(y), \tag{2.36}$$

substitute into (2.34), multiply by $p_m(y)$ and integrate over L_b. The result is

$$\sum_{n=0}^{N} b_n M_{mn} = G_{m0}, \tag{2.37}$$

where

$$M_{mn} = \sum_{r=1}^{\infty} \alpha_r h G_{mr} G_{nr}, \tag{2.38}$$

with

$$G_{mn} = \int_{L_b} p_m(y)\psi_n(y)\,\mathrm{d}y, \tag{2.39}$$

and we have assumed that the eventual choice of $p_n(y)$ ensures the convergence of the series defining M_{mn}. Then A^{-1} is approximated by

$$\sum_{n=0}^{N} b_n G_{n0} = A_u^{-1} \leqslant A^{-1}, \tag{2.40}$$

since

$$\sum_{n=1}^{\infty} \alpha_n h \left(\int_{L_b} g(y)\psi_n(y)\,\mathrm{d}y \right)^2 \geqslant 0, \quad \forall g. \tag{2.41}$$

It follows that

$$A_l \leqslant A \leqslant A_u, \tag{2.42}$$

with corresponding complementary bounds on $|R|$, $|T|$ from (2.25), (2.27).

It remains to choose the functions $u_n(y)$, $p_n(y)$ appropriate to the particular configuration being considered.

2.1. *The surface-piercing barrier*

We assume a single surface-piercing barrier of length a so that L_b is $(0, a)$ and L_g is (a, h). An analysis of the flow close to the edge of the barrier reveals that $u(y) \sim (y-a)^{-1/2}$ as $y \to a^+$ and since $\phi_y = 0$ on $y = h$, ϕ and hence $u(y) \propto \phi_x|_{x=0}$ can be continued as an even function of y across $y = h$. Therefore the even continuous function $\{(h-a)^2 - (h-y)^2\}^{1/2} u(y)$ can be expanded in (a, h) in a complete set of even Chebychev polynomials. Thus, anticipating subsequent simplifications, we choose

$$u_m(y) = \frac{2(-1)^m}{\pi \{(h-a)^2 - (h-y)^2\}^{1/2}} T_{2m}\left(\frac{h-y}{h-a}\right), \quad y \in [a, h], \tag{2.43}$$

where $T_n(x) = \cos n\theta$, $x = \cos\theta$, whence

$$F_{mn} = \int_a^h u_m(y)\psi_n(y)\,\mathrm{d}y$$
$$= \frac{2N_n^{-1/2}(-1)^m}{\pi}\int_0^1 \frac{T_{2m}(t)}{(1-t^2)^{1/2}}\cos\{k_n(h-a)t\}\,\mathrm{d}t,$$

after putting $h - y = (h-a)t$,

$$= N_n^{-1/2} J_{2m}\{k_n(h-a)\} \tag{2.44}$$

(Erdélyi *et al.* 1954, §1.10, equation 2) whilst

$$F_{m0} = (-1)^m N_0^{-1/2} I_{2m}\{k(h-a)\}, \tag{2.45}$$

where J_n, I_n are the Bessel function and modified Bessel function of order n respectively. Thus from (2.27)

$$K_{mn} = \sum_{r=1}^{\infty}(N_r\alpha_r h)^{-1} J_{2m}\{k_r(h-a)\}\, J_{2n}\{k_r(h-a)\}. \tag{2.46}$$

In choosing the functions $p_n(y)$ we bear in mind the free surface condition (2.5) and the behaviour $p(y) \sim (a-y)^{1/2}$ as $y \to a^-$ deduced from a consideration of the flow field in the neighbourhood of $y = a$. Thus we choose $p_n(y)$ to satisfy

$$Kp_n(y) + p_n'(y) = 0, \quad y = 0, \tag{2.47}$$
$$p_n(y) = 0, \quad y = a. \tag{2.48}$$

It is convenient to introduce a reduced function $\hat{p}_n(y)$ defined by

$$\hat{p}_n(y) = p_n(y) - K\int_y^a p_n(y)\,\mathrm{d}t, \tag{2.49}$$

from which it is easily verified that

$$\hat{p}_n'(y) = 0, \quad y = 0, \tag{2.50}$$
$$\hat{p}_n(y) = 0, \quad y = a. \tag{2.51}$$

Now

$$G_{mn} = \int_0^a p_m(y)\psi_n(y)\,\mathrm{d}y$$
$$= N_n^{-1/2}\int_0^a p_m(y)\cos k_n(h-y)\,\mathrm{d}y$$
$$= N_n^{-1/2}\cos k_n h\int_0^a \hat{p}_m(y)\cos k_n y\,\mathrm{d}y,$$

after integration by parts. Condition (2.50) permits the extension of $\hat{p}_n(y)$ into $(-a, 0)$ as an even function of y and taking into account (2.51) and subsequent simplifications we write

$$\hat{p}_m(y) = \frac{2(-1)^m}{\pi(2m+1)ah}(a^2-y^2)^{1/2} U_{2m}\left(\frac{y}{a}\right), \quad y \in [0, a], \tag{2.52}$$

where $U_n(x) = \sin(n+1)\theta/\sin\theta$, $x = \cos\theta$ is the orthogonal Chebychev polynomial of the second kind. It follows that

$$G_{mn} = \frac{2(-1)^m N_n^{-1/2} \cos k_n h}{\pi(2m+1)ah} \int_0^a (a^2 - y^2)^{1/2} U_m(y/a) \cos k_n y \, \mathrm{d}y$$
$$= N_n^{-1/2} \frac{\cos k_n h}{k_n h} J_{2m+1}(k_n a) \tag{2.53}$$

(Erdélyi *et al.* 1954, §1.10, equation 3), whilst

$$G_{m0} = (-1)^m N_0^{-1/2} \frac{\cosh kh}{kh} I_{2m+1}(ka). \tag{2.54}$$

Note that the $k_n h$ factor in the denominator of (2.53) ensures that the sum in (2.38) converges and we have

$$M_{mn} = \sum_{r=1}^{\infty} \alpha_r h (k_r h)^{-2} N_r^{-1} \cos^2 k_r h J_{2m+1}(k_r a) J_{2n+1}(k_r a). \tag{2.55}$$

2.2. *The submerged bottom-standing barrier*

We assume a submerged barrier extending upwards a distance $h-b$ from the bottom, so that L_b is $b < y < h$ whilst L_g is $0 < y < b$.

We have

$$F_{mn} = \int_0^b u_m(y) \cos k_n(h-y) \, \mathrm{d}y$$
$$= N_n^{-1/2} \cos k_n h \int_0^b \hat{u}_m(y) \cos k_n y \, \mathrm{d}y, \tag{2.56}$$

where

$$\hat{u}_m(y) = u_m(y) - K \int_y^b u_m(t) \, \mathrm{d}t, \tag{2.57}$$

and

$$\hat{u}'_m(y) = 0, \quad y = 0. \tag{2.58}$$

Choosing

$$\hat{u}_m(y) = \frac{2(-1)^m}{\pi(b^2 - y^2)^{1/2}} T_{2m}\left(\frac{y}{b}\right), \quad y \in [0, b], \tag{2.59}$$

now gives

$$F_{mn} = N_n^{-1/2} \cos k_n h \, J_{2m}(k_n b), \tag{2.60}$$

with

$$F_{m0} = (-1)^m N_0^{-1/2} \cosh kh \, I_{2m}(kb), \tag{2.61}$$

whilst

$$G_{mn} = \int_b^h p_m(y) \psi_n(y) \, \mathrm{d}y, \tag{2.62}$$

where we choose

$$p_m(y) = \frac{2(-1)^m \left\{(h-b)^2 - (h-y)^2\right\}^{1/2}}{\pi(2m+1)(h-b)h} U_{2m}\left(\frac{h-y}{h-b}\right), \quad y \in [b, h], \tag{2.63}$$

yielding

$$G_{mn} = \frac{N_n^{-1/2}}{k_n h} J_{2m+1}\left\{k_n(h-b)\right\}, \tag{2.64}$$

and

$$G_{m0} = (-1)^m \frac{N_0^{-1/2}}{kh} I_{2m+1}\left\{k(h-b)\right\}. \tag{2.65}$$

2.3. *A barrier with a single gap*

We assume L_g is $a < y < b$ so that L_b is $(0,a) \cup (b,h)$ and two separate expansions will be needed for the pressure difference in this case.

The square-root singularities at the edges $y = a, b$ suggest that we expand the continuous function $(y-a)^{1/2}(b-y)^{1/2}u(y)$ in terms of the complete Chebychev polynomials T_n over $[a,b]$. Thus we write

$$u_m(y) = \frac{1}{\pi(y-a)^{1/2}(b-y)^{1/2}} T_m\left(\frac{2y-a-b}{b-a}\right), \quad y \in [a,b], \tag{2.66}$$

whence

$$\begin{aligned} F_{mn} &= N_n^{-1/2} \int_a^b \frac{T_m((2y-a-b)/(b-a))}{\pi(y-a)^{1/2}(b-y)^{1/2}} \cos k_n(h-y)\,\mathrm{d}y \\ &= \frac{N_n^{-1/2}}{\pi} \int_{-1}^{1} \frac{T_m(t)}{(1-t^2)^{1/2}} \cos k_n(h-c-dt)\,\mathrm{d}t, \end{aligned}$$

where $c = \frac{1}{2}(a+b)$, $d = \frac{1}{2}(b-a)$. So

$$F_{mn} = N_n^{-1/2} \cos(k_n(h-c) - \tfrac{1}{2}m\pi) J_m(k_n d) \tag{2.67}$$

(Gradshteyn & Ryzhik 1981, p.402) so that

$$F_{2m,n} = (-1)^m N_n^{-1/2} \cos k_n(h-c)\, J_{2m}(k_n d), \tag{2.68}$$

$$F_{2m+1,n} = (-1)^{m+1} N_n^{-1/2} \sin k_n(h-c)\, J_{2m+1}(k_n d), \tag{2.69}$$

whilst

$$F_{2m,0} = N_0^{-1/2} \cosh k(h-c)\, I_{2m}(kd), \tag{2.70}$$

$$F_{2m+1,0} = -N_0^{-1/2} \sinh k(h-c)\, I_{2m+1}(kd). \tag{2.71}$$

The integral equation for the pressure difference now involves two disjoint intervals $L_1 = (0,a)$ and $L_2 = (b,h)$ so we need to modify slightly the theory given in the Introduction. Thus we write (1.3) as

$$Mp \equiv M_1 p + M_2 p = \psi_0, \quad y \in L_1 \cup L_2, \tag{2.72}$$

$$(\psi_0, p) \equiv (\psi_0, p)_1 + (\psi_0, p)_2 = A^{-1}, \tag{2.73}$$

where the suffix i refers to the interval L_i, $i = 1,2$. A different set of functions is appropriate to each interval so we write

$$p(y) = \begin{cases} \displaystyle\sum_{n=0}^{N} b_n p_n^{(1)}(y), & y \in L_1 \\ \displaystyle\sum_{n=0}^{N} c_n p_n^{(2)}(y), & y \in L_2, \end{cases} \tag{2.74}$$

where here $p_n^{(i)}(y)$ are the sets of functions chosen as the expansions of $p(y)$ in the surface-piercing ($i = 1$) and bottom-standing ($i = 2$) problems. Substitution of (2.74) into (2.72), multiplication by first $p_m^{(1)}(y)$ and then $p_m^{(2)}(y)$ and integration over L_1 and L_2 respectively, gives

$$\sum_{n=0}^{N} \left\{ b_n M_{mn}^{(1,1)} + c_n M_{mn}^{(1,2)} \right\} = (p_m^{(1)}, \psi_0)_1 = P_{m0}^{(1)}, \tag{2.75}$$

$$\sum_{n=0}^{N} \left\{ b_n M_{mn}^{(2,1)} + c_n M_{mn}^{(2,2)} \right\} = (p_m^{(2)}, \psi_0)_2 = P_{m0}^{(2)}, \tag{2.76}$$

whilst

$$\begin{aligned} A_u^{-1} &= \sum_{n=0}^{N} \left\{ b_n (p_n^{(1)}, \psi_0)_1 + c_n (p_n^{(2)}, \psi_0)_2 \right\} \\ &= \sum_{n=0}^{N} \left\{ b_n P_{n0}^{(1)} + c_n P_{n0}^{(2)} \right\}. \end{aligned} \tag{2.77}$$

Here

$$M_{mn}^{(i,j)} = (M_j p_n^{(j)}, p_m^{(i)})_i = \sum_{r=1}^{\infty} \alpha_r h P_{mr}^{(i)} P_{nr}^{(j)}, \quad i, j = 1, 2, \tag{2.78}$$

whilst

$$P_{mn}^{(i)} = (p_n^{(i)}, \psi_m)_i = \int_{L_i} p_n^{(i)}(y) \psi_m(y) \, dy, \quad i = 1, 2. \tag{2.79}$$

Note that $P_{mn}^{(i)}$, $i = 1, 2$ are identical to (2.53) and (2.64) occurring in the single surface-piercing barrier and bottom-standing barrier problems respectively.

2.4. *The totally submerged barrier*

We assume that the barrier occupies $a < y < b$ so that L_b is (a, b) and L_g is $(0, a) \cup (b, h)$ and so two separate expansions are needed for the velocity in each of the gaps.

The only requirement on $p_n(y)$, $y \in (a, b)$, is that $p(y) \sim (y - a)^{1/2}$ as $y \to a^+$, $p(y) \sim (b - y)^{1/2}$ as $y \to b^-$, and we need to use the full set U_n of second-kind Chebychev polynomials here. Thus we choose

$$p_m(y) = \frac{(y-a)^{1/2}(b-y)^{1/2}}{\pi(m+1)dh} U_m\left(\frac{2y-a-b}{b-a}\right), \quad y \in [a, b], \tag{2.80}$$

whence

$$\begin{aligned} G_{mn} &= N_n^{-1/2} \int_a^b \frac{(y-a)^{1/2}(b-y)^{1/2}}{\pi(m+1)dh} U_m\left(\frac{2y-a-b}{b-a}\right) \cos k_n(h-y) \, dy \\ &= \frac{dN_n^{-1/2}}{\pi(m+1)h} \int_{-1}^{1} (1-t^2)^{1/2} U_m(t) \cos k_n(h-c-dt) \, dt, \end{aligned}$$

where $c = h - \frac{1}{2}(a + b)$, $d = \frac{1}{2}(b - a)$ as in the previous problem. After some manipulation we find

$$G_{mn} = N_n^{-1/2}(k_n d)^{-1} \cos(k_n(h-c) - \tfrac{1}{2}m\pi) J_{m+1}(k_n d), \tag{2.81}$$

and so

$$G_{2m,n} = (-1)^m N_n^{-1/2} \frac{\cos k_n(h-c)}{k_n h} J_{2m+1}(k_n d), \tag{2.82}$$

$$G_{2m+1,n} = (-1)^{m+1} N_n^{-1/2} \frac{\sin k_n(h-c)}{k_n h} J_{2m+2}(k_n d), \tag{2.83}$$

whilst

$$G_{2m,0} = N_0^{-1/2} \frac{\cosh k(h-c)}{kh} I_{2m+1}(kd), \tag{2.84}$$

$$G_{2m+1,0} = -N_0^{-1/2} \frac{\sinh k(h-c)}{kh} I_{2m+2}(kd). \tag{2.85}$$

For the velocity formulation, we choose

$$u(y) = \begin{cases} \sum_{n=0}^{N} a_n u_n^{(1)}(y), & y \in L_1 \\ \sum_{n=0}^{N} b_n u_n^{(2)}(y), & y \in L_2, \end{cases} \tag{2.86}$$

where here $u_n^{(i)}(y)$, are the sets of functions chosen for the expansion of $u(y)$ in the surface-piercing ($i = 2$) and bottom-standing ($i = 1$) problems. Following the procedure used for the disjoint intervals in the gap between two barriers, we finally obtain

$$\sum_{n=0}^{N} \left\{ a_n K_{mn}^{(1,1)} + b_n K_{mn}^{(1,2)} \right\} = U_{m0}^{(1)}, \tag{2.87}$$

$$\sum_{n=0}^{N} \left\{ a_n K_{mn}^{(2,1)} + b_n K_{mn}^{(2,2)} \right\} = U_{m0}^{(2)}, \tag{2.88}$$

with

$$A_l = \sum_{n=0}^{N} \left\{ a_n U_{n0}^{(1)} + b_n U_{n0}^{(2)} \right\}. \tag{2.89}$$

Here

$$K_{mn}^{(i,j)} = \sum_{r=1}^{\infty} (\alpha_r h)^{-1} U_{mr}^{(i)} U_{nr}^{(j)}, \tag{2.90}$$

whilst

$$U_{mn}^{(i)} = \int_{L_i} u_n^{(i)}(y) \psi_m(y)\, dy, \tag{2.91}$$

Note that $U_{mn}^{(i)}$, $i = 1, 2$ are identical to (2.60) and (2.44) occurring in the single bottom-standing barrier of length $h - a$ and surface-piercing barrier of length b problems respectively.

3. Scattering by two identical barriers with gaps

As a second illustration of the method we consider the scattering of an obliquely incident wave in finite depth h by two identical vertical barriers with gaps occupying the planes $x = \pm b$. From symmetry considerations it is possible to express the time-independent potential $\phi(x, y)$ defined by (2.1) in the form

$$\phi(x,y) = \phi^s(x,y) + \phi^a(x,y), \tag{3.1}$$

where

$$\left.\begin{array}{l} \phi^s(x,y) = \phi^s(-x,y) \\ \phi^a(x,y) = -\phi^a(-x,y) \end{array}\right\}. \tag{3.2}$$

We need only consider $x \geqslant 0$, using (3.2) to extend the solution into $x \leqslant 0$. Then $\phi^{s,a}$ satisfy (2.2) to (2.6) as before but in addition we have

$$\frac{\partial\phi^s}{\partial x}(0,y) = \phi^a(0,y) = 0, \tag{3.3}$$

from (3.2). An appropriate form for ϕ^s is

$$\phi^s(x,y) = (\mathrm{e}^{-\mathrm{i}\alpha(x-b)} + R^s\mathrm{e}^{\mathrm{i}\alpha(x-b)})\psi_0(y) + \sum_{n=1}^{\infty} A_n \mathrm{e}^{-\alpha_n(x-b)}\psi_n(y), \quad x > b, \tag{3.4}$$

$$\phi^s(x,y) = B_0^s \cos\alpha x\, \psi_0(y) + \sum_{n=1}^{\infty} B_n^s \cosh\alpha_n x\, \psi_n(y), \quad 0 < x < b, \tag{3.5}$$

whilst $\phi^a(x,y)$ differs only in replacing R^s by R^a in (3.4) and $\cos\alpha x$, $\cosh\alpha_n x$ by $\sin\alpha x$, $\sinh\alpha_n x$ in (3.5) with different coefficients B_0^a, B_n^a.

It follows that

$$R = \tfrac{1}{2}(R^s + R^a), \quad T = \tfrac{1}{2}(R^s - R^a), \tag{3.6}$$

where $|R^{s,a}| = 1$. A matching of $\phi^{s,a}$ and $\phi_x^{s,a}$ and use of the Galerkin method as before produces the following changes. The K_{mn} in (2.27) now includes the factor $(1 + \coth\alpha_r b)$ for ϕ^s and $(1 + \tanh\alpha_r b)$ for ϕ^a whilst A in (2.24) is replaced by

$$A^s = \frac{\mathrm{i}\alpha h(R^s\mathrm{e}^{2\mathrm{i}\alpha b} - 1)}{(R^s\mathrm{e}^{2\mathrm{i}\alpha b} + 1) + \mathrm{i}(R^s\mathrm{e}^{2\mathrm{i}\alpha b} - 1)\cot\alpha b}, \tag{3.7}$$

in the symmetric case and by

$$A^a = \frac{\mathrm{i}\alpha h(R^a\mathrm{e}^{2\mathrm{i}\alpha b} - 1)}{(R^a\mathrm{e}^{2\mathrm{i}\alpha b} + 1) - \mathrm{i}(R^a\mathrm{e}^{2\mathrm{i}\alpha b} - 1)\tan\alpha b}, \tag{3.8}$$

in the antisymmetric case.

The corresponding change to M_{mn} in (2.38) is the factor $(1 + \coth\alpha_r b)^{-1}$ for ϕ^s and $(1 + \tanh\alpha_r b)^{-1}$ for ϕ^a.

4. The sloshing problem

The same technique can be applied to determine the sloshing frequencies in a rectangular tank with a baffle consisting of a thin vertical barrier with an arbitrary number of gaps. A particular case of this problem was considered by Evans & McIver (1987). We assume that the baffle separates the tank into two regions, $-b \leqslant x \leqslant 0$ and $0 \leqslant x \leqslant c$, $b + c = d$, and the tank occupies $-b \leqslant x \leqslant c$, $0 \leqslant z \leqslant f$, $0 \leqslant y \leqslant h$. Then the only changes required are that the term $\mathrm{e}^{\mathrm{i}lz}$ in (2.1) is replaced by $\cos lz$ where $lf = m\pi$, $m = 0, 1, 2, \ldots$ and that (2.12), (2.13) are replaced by

$$\phi(x,y) = A_0 \cos\alpha(x+b)\psi_0(y) + \sum_{n=1}^{\infty} A_n \cosh\alpha_n(x+b)\psi_n(y), \quad -b \leqslant x \leqslant 0, \tag{4.1}$$

and

$$\phi(x,y) = B_0 \cos\alpha(c-x)\psi_0(y) + \sum_{n=1}^{\infty} B_n \cosh\alpha_n(c-x)\psi_n(y), \quad 0 \leqslant x \leqslant c. \tag{4.2}$$

The result of repeating the development leading to (2.27), (2.38) is to include a factor $(\coth\alpha_r b + \coth\alpha_r c)$ in K_{mn} and $(\coth\alpha_r b + \coth\alpha_r c)^{-1}$ in M_{mn} while A is replaced by

$$A^{(s)} = \frac{\alpha h \sin\alpha b \sin\alpha c}{\sin\alpha d}. \tag{4.3}$$

The problem is reduced to determining $A^{(s)}$ and hence, for given b, c, f, solving for $\alpha = \alpha^{(p)}$, $p = 1, 2, \ldots$ and determining the sloshing frequencies ω_{mp} from

$$\omega^2 = gk \tanh kh \tag{4.4}$$

where $k^2 = l^2 + \alpha^2$ with $l = m\pi/f$, $m = 0, 1, 2, \ldots$.

5. Results

Common to all problems considered in this paper is the computation of a quantity related to global quantities of interest for the particular problem.

In each case we have established complementary bounds for A but only (2.22) translates into complementary bounds for $|R|$. However, we shall obtain such close bounds for A that the results for R, R^s, R^a or the sloshing frequencies in (2.25), (3.7), (3.8) or (4.3) will be equally close if not complementary.

The numerical procedure we use is common to all the problems considered. To fix ideas we illustrate in detail what is involved in the case of the single surface-piercing barrier. In order to solve for A_l, A_u we must first compute K_{mn}, M_{mn} given by (2.46) and (2.55) for $m, n = 0, \ldots, N$. Each entry consists of an infinite sum which is evaluated by truncation. It is noted that, as in all the problems considered in this paper, the infinite series decays like $O(1/r^2)$ and thus many terms are needed to yield accurate entries for K_{mn}, M_{mn}. For a given truncation size M, say, it is possible to significantly increase the accuracy of these entries with little extra effort. Using the well-known fact $k_r h \sim r\pi$ as $r \to \infty$ and the asymptotic behaviour of $J_n(x)$ for large x it is readily shown that, as $r \to \infty$,

$$(\alpha_r h)^{-1} F_{mr} F_{nr} = \frac{2(-1)^{m+n}}{\pi^3(1-a/h)} s_r + O(1/r^4), \tag{5.1}$$

and

$$\alpha_r h G_{mr} G_{nr} = \frac{2(-1)^{m+n}}{\pi^3 a/h} s_r + O(1/r^4), \tag{5.2}$$

where

$$s_r = (1 - \sin 2r\pi a/h)/r^2. \tag{5.3}$$

So, in writing

$$K_{mn} = \sum_{r=1}^{M} (\alpha_r h)^{-1} F_{mr} F_{nr} + \frac{2(-1)^{m+n}}{\pi^3(1-a/h)} S, \tag{5.4}$$

$$M_{mn} = \sum_{r=1}^{M} \alpha_r h G_{mr} G_{nr} + \frac{2(-1)^{m+n}}{\pi^3 a/h} S, \tag{5.5}$$

N	$a/h = 0.1$ A_l ($\leqslant A \leqslant$)	A_u	$a/h = 0.5$ A_l ($\leqslant A \leqslant$)	A_u	$a/h = 0.9$ A_l ($\leqslant A \leqslant$)	A_u
0	2.551449	10.225129	2.6402985	2.6424255	0.7358405	0.7451010
1	10.020323	10.225129	2.6420460	2.6420502	0.7358405	0.7360358
2	10.223271	10.225129	2.6420501	2.6420502	0.7358405	0.7358475
3	10.224647	10.225129	2.6420501	2.6420502	0.7358405	0.7358407
4	10.225108	10.225129	2.6420501	2.6420502	0.7358405	0.7358405
5	10.225129	10.225129	2.6420501	2.6420502	0.7358405	0.7358405

TABLE 1. A_l, A_u for a surface-piercing barrier with $ka = 0.5$, $\theta = 0$

with

$$S = \sum_{r=M+1}^{\infty} s_r, \tag{5.6}$$

we are including the leading-order asymptotic behaviour of the original series and neglecting only the $O(1/M^4)$ terms. This contrasts with a straightforward truncation of the series where the $O(1/M^2)$ terms are ignored. We use 10^6 terms in the computation of S; this is a relatively quick calculation to perform owing to the simplicity of s_r in (5.3). It is noted that S does not depend upon m, n, or the wavenumber k, and thus need only be calculated once for a particular geometry. A similar approach is used for all the problems considered in this paper to improve the convergence of the infinite sums. The number of terms required to guarantee a given accuracy in the elements K_{mn}, M_{mn} varies with the geometry being considered. Thus, for typical geometries, not involving very small gaps or very small barriers where almost total reflection or total transmission occurs, the above procedure enables 8-figure accuracy in K_{mn}, M_{mn} to be achieved with $M \sim 500$ and the values obtained for A_l and A_u from solving (1.9) and (1.16) differ only in the seventh figure for $N \sim 5$. In the extreme cases mentioned above it is necessary to choose larger values of M and N for similar accuracy but it will turn out that in these cases one or other of the formulations, but not both, converges to the solution for values of N as low as 0 or 1. Following extensive numerical experiments the values of $M = 500$, $N = 4$ were chosen to produce the graphs in all cases, it not being possible to distinguish the curves derived from the two formulations, and a value of $M = 2000$ was used in producing the numbers in the tables shown. For a discussion of the choice of N, M to ensure convergence of A in a related problem, see Wu (1973).

Having determined K_{mn}, M_{mn}, and solved the systems (1.9) and (1.16) for a given truncation size N, A_l, A_u are computed from (1.10) and (1.17). The results for the single surface-piercing barrier are summarized in table 1 which shows clearly the accuracy of the numerical method.

The table shows the values of A_l, A_u, respectively the lower and upper bounds on A, when the barrier is submerged to depths $a/h = 0.1$, 0.5, 0.9 in normally incident waves with non-dimensionalized wavenumber $ka = 0.5$ in each case. The truncation size, N, is increased from 0 (equivalent to a one-term variational approximation) until the discrepancy between the upper and lower bounds, A_l, A_u, is such that 7-figure accuracy is obtained in A. From (2.25), (2.26) this translates into complementary bounds on the reflection and transmission coefficients with the same order of accuracy in them.

It is clear from the table that $N = 5$ is sufficient to determine A to 7 significant figures at this typical wavelength for both a short barrier ($a/h = 0.1$) or for a small

N	$b/h = 0.1$ A_l ($\leqslant A \leqslant$) A_u		$b/h = 0.4$ A_l ($\leqslant A \leqslant$) A_u		$b/h = 0.8$ A_l ($\leqslant A \leqslant$) A_u	
0	3.0956544	3.4337470	3.2771580	3.2804826	20.855972	31.981896
1	3.0956752	3.0999419	3.2798358	3.2798363	31.947507	31.981895
2	3.0956752	3.0957406	3.2798362	3.2798362	31.981683	31.981895
3	3.0956752	3.0956772	3.2798362	3.2798362	31.981894	31.981895
4	3.0956752	3.0956752	3.2798362	3.2798362	31.981895	31.981895

TABLE 2. A_l, A_u for a bottom-standing barrier with $kb = 0.2$, $\theta = 0$

gap beneath the barrier ($a/h = 0.9$) and indeed $N = 1$, corresponding to a 2×2 system, is sufficient to give 1% accuracy in all cases. Even more remarkable is the accuracy of the one-term ($N = 0$) approximation to A_l in the velocity formulation in the case of a small gap or the one-term approximation to A_u in the pressure difference formulation in the case of a small barrier. This can be explained as follows. In the case of a small gap, the fluid motion is determined by a rapid flow close to the edge of the barrier which is well modelled by the square-root singularity in $u_0(y)$. Equally, in the case of a small barrier the pressure difference is dominated by its behaviour at either end, which is again correctly modelled by the choice of $p_0(y)$. Similar accuracy is achieved for $\theta \neq 0$, or obliquely incident waves.

The only change required for the bottom-standing barrier is in the use of (2.60), (2.61) and (2.64), (2.65) for F_{mn} and G_{mn} in (2.27) and (2.38). Thus table 2 illustrates, for $\theta = 0$, the accuracy of the method for different gap ratios b/h above the barrier each at non-dimensionalized wavenumber $kb = 0.2$. Here $N = 4$ is sufficient to achieve 7-figure accuracy and again we see the power of a one-term approximation to A_l in the velocity approximation in the case of small gaps ($b/h = 0.1$) and a one-term approximation to A_u in the pressure difference formulation in the case of small barriers ($b/h = 0.8$)

The results shown in tables 1 and 2 were chosen to correspond to representative values of barrier geometry and incident wavenumber. Other values of wavenumber produce equally accurate results for A and hence R or T and the results are illustrated in figures 2 and 3. Figure 2 shows the variation of $|R|$ and $|T|$ with non-dimensionalized wavenumber ka in the case of the scattering of a normally incident wave by a surface-piercing barrier of length a in water depth h. For long waves, corresponding to low wavenumbers, the potential behaves like a uniform horizontal flow far from the barrier and so all the wave energy is transmitted through the gap in the barrier. For short waves, or large wavenumbers, most of the wave energy is located near the free surface so that a large proportion of the wave energy is reflected by the barrier and only a small amount will 'leak' under the barrier. The exact results for the scattering by a barrier in infinite depth due to Ursell (1947) are also shown in figure 2. It can be seen that a barrier submerged to approximately one tenth of the water depth can effectively be regarded as being in infinite-depth water at all wavenumbers.

The results for the scattering of a normally incident wave by an bottom-standing barrier extending from the bottom to a point b beneath the free surface in water of depth h are shown in figure 3 where curves of $|R|$ and $|T|$ are drawn against dimensionless wavenumber kb. For kb small, the same arguments as for the surface-piercing barrier show that near total transmission occurs, but in this case in the limit of short waves, or kb large, the waves are confined to the surface and again most of

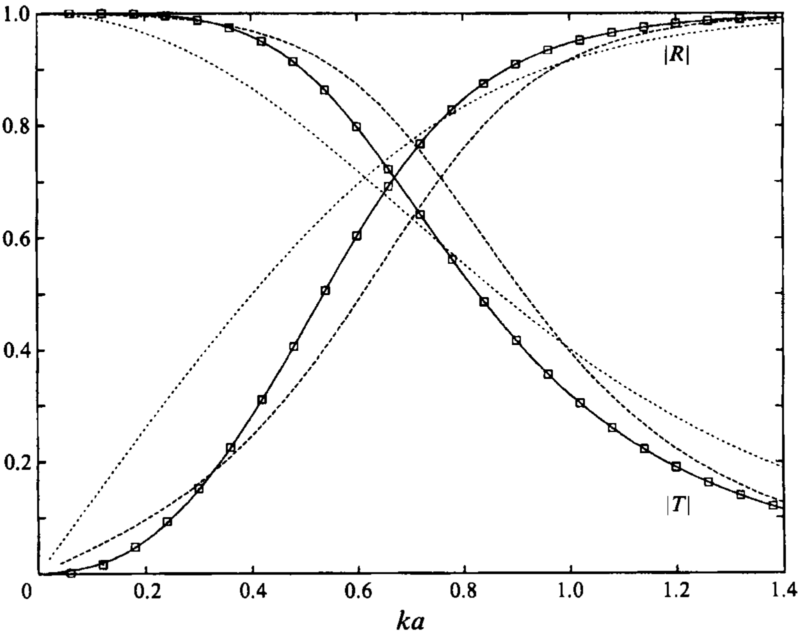

FIGURE 2. $|R|$, $|T|$ for a surface-piercing barrier in normally incident waves: ——, $a/h = 0.1$; - - - , $a/h = 0.5$; · · ·, $a/h = 0.9$; □, infinite depth.

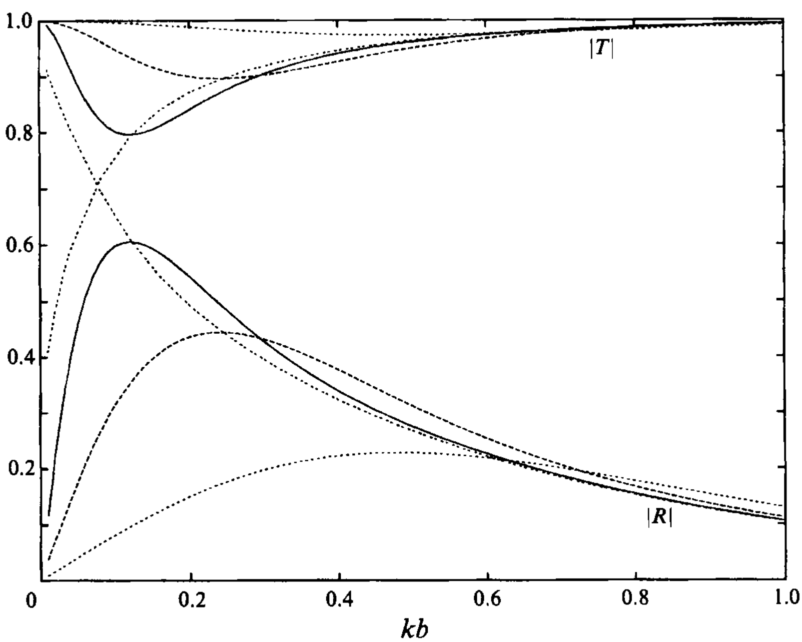

FIGURE 3. $|R|$, $|T|$ for a bottom-standing barrier in normally incident waves: ——, $b/h = 0.1$; - - -, $b/h = 0.2$; · · ·, $b/h = 0.4$; - - , infinite depth.

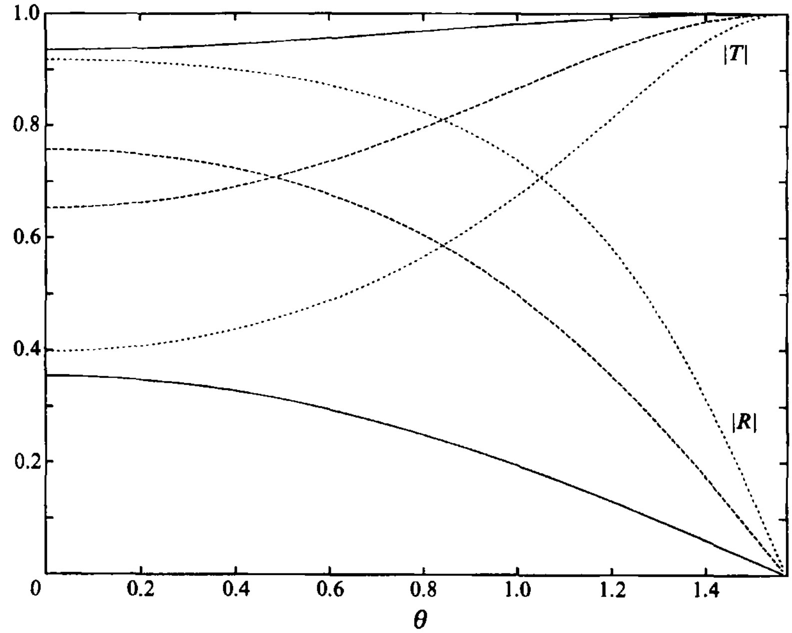

FIGURE 4. $|R|$, $|T|$ against θ for a surface-piercing barrier with $a/h = 0.5$ in obliquely incident waves: —, $ka = 0.5$; - - -, $ka = 0.8$; $\cdots$, $ka = 1.0$.

the energy is transmitted. Thus for a given b/h there exists a unique value of kb at which $|T|$ is a minimum and $|R|$ is a maximum. This contrasts with the special case of a submerged barrier in infinitely deep water solved exactly by Dean (1945). In this case very long waves are confronted with a very long barrier and total reflection occurs in the limit $kb \to 0$. This difference in behaviour for long waves means that for a barrier occupying almost all the depth, i.e. $b/h = 0.1$, then good agreement with the infinite-depth result only occurs for $kb > 0.5$.

The effect on $|R|$, $|T|$ of varying the angle of incidence is shown for the surface-piercing barrier in figure 4 and for the bottom-standing barrier in figure 5. In each case $|R|$ and $|T|$ decrease and increase monotonically (respectively) with increasing incidence angle, θ.

For a barrier with a single gap the changes necessary in the velocity formulation are slight, involving the use of (2.67) for F_{mn} whilst the pressure formulation necessitates the solution of the coupled system (2.75), (2.76) prior to computing A_u from (2.77). Despite this, the accuracy of the method is maintained and again a truncation of $N = 5$ is sufficient to assure 7-figure agreement between A_l and A_u. The same is true for the problem of the totally submerged barrier where G_{mn} is given by (2.81) and where it is the velocity formulation which requires the solution of the coupled system (2.87), (2.88) prior to computing A_l from (2.89). There seems little merit in presenting detailed tables of the A_l and A_u but it is worth emphasizing that, for small gaps and small barriers, the approximations $u(y) = u_0(y)$ and $p(y) = p_0(y)$ respectively provide extremely accurate values for R and T.

The variation of $|R|$, $|T|$ with dimensionless wavenumber for a barrier with a gap in normally incident waves ($\theta = 0$) is shown in figure 6. Here the upper part of the barrier occupies $(0, a)$, the lower part (b, h) so that the centre of the gap is at depth

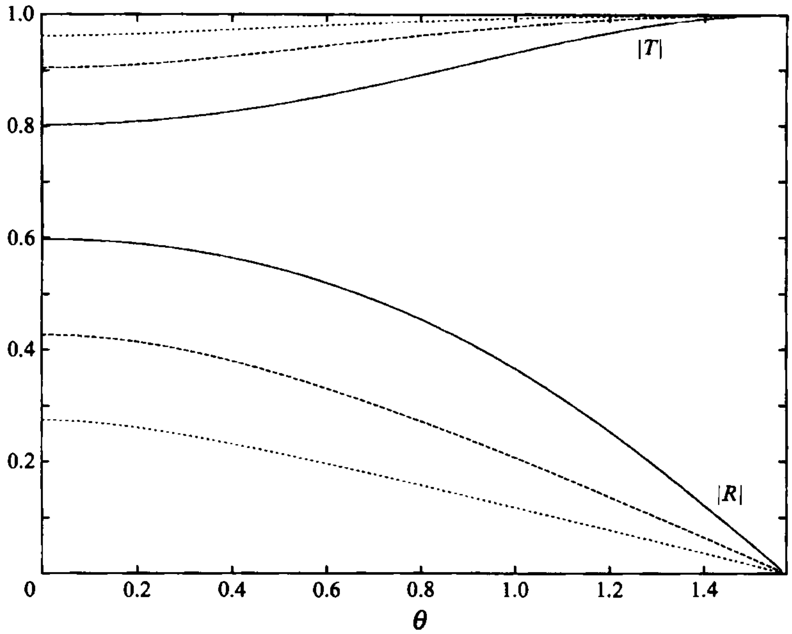

FIGURE 5. $|R|$, $|T|$ against θ for an bottom-standing barrier with $b/h = 0.1$ in obliquely incident waves: ——, $kb = 0.1$; - - - , $kb = 0.3$; $\cdots$, $kb = 0.5$.

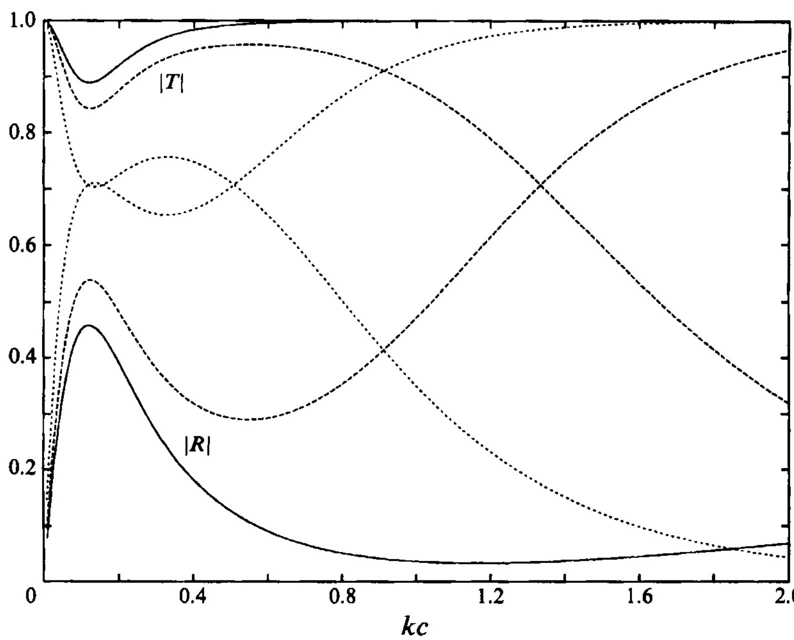

FIGURE 6. $|R|$, $|T|$ for a barrier with a gap with $c/h = 0.1$ in normally incident waves: —— , $d/c = 0.9$; - - - , $d/c = 0.5$; $\cdots$, $d/c = 0.1$.

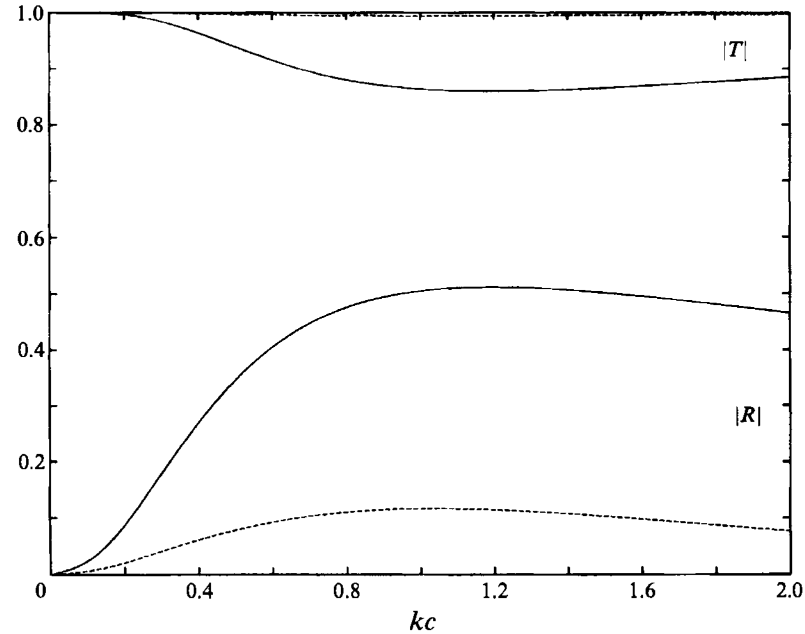

FIGURE 7. $|R|$, $|T|$ for a totally submerged barrier with $c/h = 0.1$ in normally incident waves: ——, $d/c = 0.9$; - - -, $d/c = 0.5$

$c = \frac{1}{2}(a+b)$ and the size of the gap is $2d = (b-a)$. Curves for various d/c are shown with fixed $c/h = 0.1$. As might be expected the curves of $|R|$, $|T|$ are now more complicated, manifesting behaviour of both bottom-standing and surface-piercing barriers. For example, in very long waves, $kc \ll 1$, all the energy is transmitted and in very short waves, kc large, all the energy is reflected as for the surface barrier, but in passing from long to short waves $|R|$ first reaches a maximum, as for the bottom-standing barrier and then at shorter waves a minimum, before tending monotonically to unity as $kc \to \infty$.

Figure 7 shows the behaviour of $|R|$, $|T|$ for a totally submerged barrier of length $2d$, centre at depth c with varying kc and fixed $c/h = 0.1$. The curves are less interesting than the gap case, with $|R| \to 0$ for both large and small kc as might be expected on physical grounds. Once again in each case the effect of increasing θ is to reduce $|R|$ and increase $|T|$.

The scattering by two identical barriers with gaps requires only a slight modification to what has gone before, notably in the inclusion of extra hyperbolic terms in the infinite systems involving K_{mn}, M_{mn}, but the complicated nature of (3.7), (3.8) does not provide upper and lower bounds for $|R|$, $|T|$. Nevertheless the continuous smooth dependence of R^s, R^a on A^s, A^a and αb through (3.7), (3.8) ensures that accurate estimates of A^s, A^a result in equally accurate estimates of R and T from (3.6).

An interesting feature of two-barrier problems is the occurrence of zeros of reflection at certain frequencies and, for surface-piercing barriers, zeros of transmission also. The former phenomenon is well-known in wave transmission problems, the latter less so. As described in the introduction it was first shown to occur by Evans & Morris (1972) and subsequently confirmed by Newman (1974) in deep water and McIver (1985) in finite water depth. McIver used matched eigenfunction expansions to obtain an infinite system of equations for the unknown coefficients in the expansions of the potential and, as he pointed out, the square-root singularity at the tip of each barrier necessitated the use of quite large values of N, the truncation parameter, together with extrapolation, to achieve reasonable accuracy for R, T. To locate zeros in R or T he examined the phase of these complex coefficients, looking for values of dimensionless frequency at which the phase changes discontinuously by π radians as indicating a zero in R and T. He estimated that the frequencies so obtained had small errors in the third significant figure for the fundamental frequencies and in the fourth significant figure for the higher frequencies by choosing $N = 400$ and using extrapolation.

An alternative approach is used here in which the condition for a zero of R and T is expressed as an equation involving A^s and A^a each of which can be determined extremely accurately. Thus it is straightforward to show that the condition for $T = 0$, or $R^s = R^a$, reduces using (3.7) and (3.8) to

$$f(a/h, b/h, \alpha h) = 2\operatorname{cosec} 2\alpha b - \alpha h((A^s)^{-1} - (A^a)^{-1}) = 0, \tag{5.7}$$

whilst the condition for $R = 0$, or $R^s = -R^a$, reduces to

$$g(a/h, b/h, \alpha h) = A^s \cot \alpha b - A^a \tan \alpha b - \alpha h = 0. \tag{5.8}$$

It is clear from the simplicity of the equations (5.7) and (5.8) that accurate estimates of the zeros of R, T, just as for the values of R, T themselves, depend purely on the accuracy with which A^s, A^a can be determined.

Again, $A_l^{s,a}$, $A_u^{s,a}$ are found to be accurate over a range of geometrical and wave parameters with moderate N as in the single barrier case. For very small barrier separations or very small barriers or gaps, the same accuracy requires a larger N. Since the zeros of R and T are of interest in this problem, we will not present a table of $A_l^{s,a}$, $A_u^{s,a}$ with varying N for a set of parameters as before, but concentrate on the accuracy with which the frequencies associated with the fundamental zeros of R and T, $Ka(R_0)$ and $Ka(T_0)$ respectively, in McIver (1985) notation, can be obtained with moderate truncation size, N. Thus table 3 illustrates successive estimates of $Ka(R_0)$, $Ka(T_0)$ using a velocity and a pressure difference formulation for increasing N in the case of $a/h = 0.1$ and $b/a = 0.2$ and normally incident waves. In order to do this, we fix N and apply a root-finding procedure to f and g in (5.7) and (5.8), where $A^{s,a}$ is replaced in turn by the two approximations, $A_l^{s,a}$, $A_u^{s,a}$, derived from the velocity and the pressure difference formulations respectively, to give the two approximations to the zeros as shown in table 3. Here we have chosen $a/h = 0.1$ and $b/a = 0.2$ in order to compare our results with McIver (1985). The parameter N is successively increased from 0 until the approximations to $Ka(R_0)$ and $Ka(T_0)$, from the two methods are such that they agree to five significant figures (at $N = 8$). This accuracy is a direct consequence of the accuracy with which the upper and lower bounds $A_l^{s,a}$, $A_u^{s,a}$ to $A^{s,a}$ are found. It can be seen from table 3 that we do not have complementary bounds to $Ka(R_0)$ and $Ka(T_0)$ and we also note that the approximation derived from the pressure difference formulation provides good accuracy for a truncation size as small as $N = 1$, owing to $a/h = 0.1$ representing a small barrier and using the arguments mentioned previously. It is clear that we do not have the same accuracy as in the single barrier case. However $N = 8$ was sufficient to determine the frequencies associated with the zeros of R and T to five significant figures in all cases and results showing the non-dimensional wavenumber at which the zeros occur for varying barrier spacing are given in figure 8. It can be seen that as the spacing increases the zeros of T

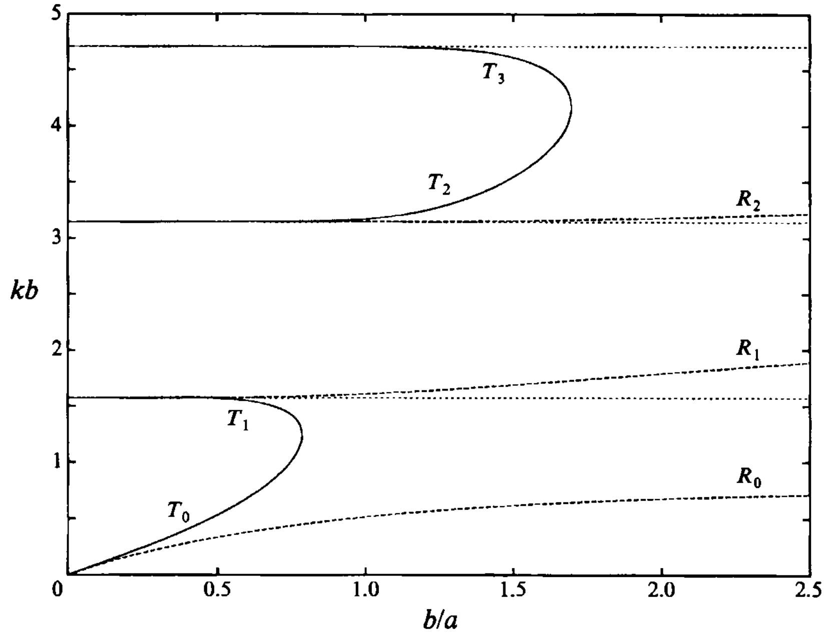

FIGURE 8. The variation of the frequencies associated with the zeros of transmission and reflection with the barrier spacing b/a for the case of $a/h = 0.1$.

	$Ka(R_0)$		$Ka(T_0)$	
N	Velocity	Pressure	Velocity	Pressure
0	0.35887	0.81381	0.37025	0.97144
1	0.68800	0.80830	0.74769	0.96130
2	0.77886	0.80830	0.89227	0.96132
3	0.80131	0.80830	0.94367	0.96133
4	0.80677	0.80830	0.95767	0.96133
5	0.80802	0.80830	0.96068	0.96133
6	0.80826	0.80830	0.96122	0.96133
7	0.80830	0.80830	0.96130	0.96133
8	0.80830	0.80830	0.96132	0.96133

TABLE 3. Approximations to the frequencies associated with the fundamental zeros of reflection and transmission for two equally submerged surface-piercing barriers, $a/h = 0.1$, $b/a = 0.2$, using a velocity and a pressure method.

coalesce in pairs. The same figure was shown in McIver (1985) who needed to employ interpolation techniques to determine these cut-off barrier spacings accurately.

The final example involves the sloshing of a liquid in a rectangular tank containing a baffle. A root-finding procedure must be applied to the equation (4.3) with $A_l^{(s)}$ and $A_u^{(s)}$ in turn approximating $A^{(s)}$ in (4.3) to provide two approximations, associated with the velocity and the pressure difference formulations respectively, to the exact sloshing frequency. The increasing accuracy of the approximations $A_l^{(s)}$, $A_u^{(s)}$ to $A^{(s)}$ as the truncation size, N, is increased is reflected in the accuracy of the estimates to the sloshing frequencies as shown in table 4. Here we have examined the fundamental sloshing frequencies when the baffle is surface piercing and close to

N	αd Velocity	Pressure
0	1.94095	2.21975
1	2.17605	2.19243
2	2.19146	2.19232
3	2.19230	2.19232
4	2.19232	2.19232

TABLE 4. Approximations to the fundamental sloshing frequencies for a surface-piercing baffle in a rectangular tank using a velocity and a pressure difference method with $a/h = 0.4$, $b/d = 0.05$, $h/d = 1.0$.

one wall ($b/d = 0.05$, $a/h = 0.4$, $h/d = 1.0$ and no cross-tank modes). Again 6-figure accuracy is obtained for $N = 4$ in this fairly extreme case. The same is true for other configurations and for totally submerged baffles. In all cases tested it was sufficient to choose $N = 5$ to achieve 6-figure accuracy to the sloshing frequencies given by the roots of (4.3).

6. Conclusion

A number of problems involving thin vertical barriers with gaps have been considered using the linear theory of water waves in finite depth. In all cases it has been shown how certain global quantities related to the solution to these problems can be determined accurately by deriving complementary bounds for the parameters related to these quantities.

Crucial to the method was the construction of singular integral equations for both the horizontal velocity of the fluid across the gap in the barrier and for the jump in dynamic pressure across the barrier, which enabled complementary bounds to be constructed. The Galerkin method, using sets of functions which accurately modelled both the velocity and the pressure difference, converted the integral equation into systems of algebraic equations resulting in extremely accurate upper and lower bounds with small values of truncation value N.

Other problems can be solved by this method. These include problems involving the partial blocking of a two-dimensional acoustic waveguide by a thin plate or two identical parallel plates or the determination of the scattering of an incident plane wave by a vertical open-ended thin circular tube partially or totally immersed in water of finite depth. A more interesting extension involves the scattering of an obliquely incident wave by a periodic array of one or two rows of thin barriers extending throughout the water depth. The present theory needs to be modified to accomodate the complex-valued kernel $K(y, t)$ arising in this case but this presents no real difficulties. Results for both of these latter problems will be published separately.

One of us, R.P., wishes to acknowledge the receipt of SERC research studentship number 92004667.

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